

CONSTRUCTING AN EXPANDING METRIC FOR DYNAMICAL SYSTEMS IN ONE COMPLEX VARIABLE

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ABSTRACT. We describe a rigorous computer algorithm for attempting to construct an explicit, discretized metric for which a polynomial map $f: \mathbb{C} \rightarrow \mathbb{C}$ is expansive on a neighborhood of the Julia set, J . We show construction of such a metric proves the map is hyperbolic. We also examine the question of whether the algorithm can be improved, and the related question of how to build a metric close to euclidean. Finally, we give several examples generated with our implementation of this algorithm.

1. INTRODUCTION

Our main concern in this paper is to develop and use *rigorous* computer investigations to study the dynamics of polynomial maps of \mathbb{C} of degree $d > 1$; for example, $f(z) = z^2 + c$. For complex polynomial maps, the invariant set of interest is the *Julia set*, J . The Julia set can be defined as the topological boundary of the set of points with bounded orbits. Intuitively, J is precisely where the chaotic dynamics occurs. For example, f is topologically transitive on J ; also, J is non-empty and perfect (see for instance [11]).

The hyperbolic polynomials are a large class of maps with chaotic dynamics, but whose stability properties make them amenable to computer study. A polynomial map f is called *hyperbolic*, or *expansive*, if f is uniformly expanding on some neighborhood of J , with respect to some riemannian metric. Uniform expansion forces some dynamical rigidity. For example, hyperbolicity of a polynomial map f implies *structural stability*, i.e., in a neighborhood of f in parameter space, the dynamical behavior is of constant topological conjugacy type. Further, if f is hyperbolic then the orbit of every point in the complement of J tends to either some attracting periodic orbit or infinity. In fact, a polynomial map f is hyperbolic if and only if the orbit of every *critical* point of f tends to an attracting periodic orbit or infinity. Thus the fate of the critical points provides a straightforward test for hyperbolicity. In this paper we develop an alternate test for hyperbolicity which produces more explicit information about the dynamics of any given map.

We begin with the work of [9] as a foundation. There we described a rigorous algorithm (and its implementation) for constructing a neighborhood \mathcal{B} of J , and a graph Γ which models the dynamics of f on \mathcal{B} . In this paper, we further our program by developing a rigorous algorithm for attempting to construct a metric in which a given f is expansive, by some uniform factor $L > 1$, on the neighborhood \mathcal{B} . We show that successful construction of such a metric proves that f is hyperbolic.

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In addition, we analyze both some limitations of and possible improvements to our basic algorithm. We show, via counterexamples, that a significantly simpler algorithm will not suffice. On the other hand, we do present an enhancement to our algorithm, to build a metric closer to euclidean.

One motivation for improving the algorithm and the metric which it builds is that in [8], we use this one dimensional algorithm as part of a computer-assisted proof of hyperbolicity of polynomial diffeomorphisms of \mathbb{C}^2 , Hénon mappings in particular. The one dimensional algorithm is used in such a way that that the closer to euclidean we can build the metric, the more likely we are to succeed at proving hyperbolicity in \mathbb{C}^2 . Thus improving the current methods could lead to more interesting examples of proven hyperbolic polynomial diffeomorphisms of \mathbb{C}^2 .

We implemented the algorithms of this paper in a program *Hypatia*,¹ and at the conclusion of the paper, we give several examples of metrics constructed which establish hyperbolicity of some polynomial maps of degrees two and three.

1.1. Statement of main results. In [9] we described the *box chain construction*, for building a directed graph Γ representing f on some neighborhood \mathcal{B} of J . A similar approach is in the body of work described in the survey [12] (cf [5, 6, 16, 19]). In this paper we need only:

Definition 1.1. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a directed graph, with vertex set $\mathcal{V} = \mathcal{V}(\Gamma) = \{B_k\}_{k=1}^N$, a finite collection of closed boxes in \mathbb{C} , having disjoint interiors, and such that the union of the boxes $\mathcal{B} = \mathcal{B}(\Gamma) = \cup_{k=1}^N B_k$ contains J . Suppose there is a $\delta > 0$ such that Γ contains an edge from B_k to B_j if the image $f(B_k)$ intersects a δ -neighborhood of B_j , *i.e.*,

$$\mathcal{E} \supset \{(k, j) : f(B_k) \cap \mathcal{N}(B_j, \delta) \neq \emptyset\}.$$

Further, assume Γ is strongly connected, *i.e.*, for each pair of vertices B_k, B_j , there is a path in Γ from B_k to B_j , and vice-versa. Then we call Γ a *box chain model of f on J* and \mathcal{B} a *box Julia set*.

For the remainder of this paper, Γ will denote a box chain model of f on J , for some polynomial map $f : \mathbb{C} \rightarrow \mathbb{C}$ of degree greater than one.

Remark 1. A box chain model of f on J satisfies the definition of a *symbolic image of f* , given by Osipenko in [15].

Given a box chain model Γ , our first goal in this paper is to define a version of hyperbolicity for Γ which is checkable by computer, and which implies hyperbolicity for f . Our discrete version of hyperbolicity is parallel to the following standard definition, from [11]: a polynomial $f : \mathbb{C} \rightarrow \mathbb{C}$ is called *hyperbolic* if there exists a riemannian metric μ , on some neighborhood \mathcal{B} of J , and an *expansion constant* $\lambda > 1$, such that the derivative $D_z f$ at every point z in \mathcal{B} satisfies $|D_z f(\mathbf{v})|_\mu \geq \lambda |\mathbf{v}|_\mu$, for every vector \mathbf{v} in the tangent space $T_z \mathbb{C}$.

First, we introduce the following discrete version of a riemannian metric.

Definition 1.2. Let $\mathcal{V} = \{B_k\}_{k=1}^N$ be a finite collection of closed boxes in \mathbb{C} , with disjoint interiors. Let $\Phi : (\sqcup_{k=1}^N B_k) \rightarrow \mathbb{R}^+$ be a box constant function, *i.e.*, for some set of positive constants $\{\varphi_k\}_{k=1}^N$, we have $\Phi(z) = \varphi_k$ if $z \in B_k$. Then Φ times euclidean induces a metric on $T\mathcal{V} = \sqcup_{k=1}^N TB_k$. That is, for any $B_k \in \mathcal{V}$, $z \in B_k$,

¹To obtain a copy of this unix program, write to the author.

and $\mathbf{v} \in T_z B_k$, let $|\mathbf{v}|_k := \varphi_k |\mathbf{v}|$. Let $|\cdot|_\Phi$ be the induced norm on $\sqcup_{k=1}^N T B_k$. We call Φ a *box metric*, and say φ_k is a *handicap* for B_k .

Note that if $z \in B_k \cap B_j$, then $|\mathbf{v}|_\Phi = |\mathbf{v}|_k$ if we are considering $\mathbf{v} \in T_z B_k$, but $|\mathbf{v}|_\Phi = |\mathbf{v}|_j$ if we are considering $\mathbf{v} \in T_z B_j$. We justify the choice of the name “handicap” a few paragraphs below.

Now we present our discretized version of hyperbolicity.

Definition 1.3. Call $\Gamma = (\mathcal{V}, \mathcal{E})$ *box expansive* if there exists a box metric Φ on \mathcal{V} and a *box expansion constant* $L > 1$ such that for all $(k, j) \in \mathcal{E}$, $z \in B_k$, and $\mathbf{v} \in T_z B_k$, we have $|D_z f(\mathbf{v})|_j \geq L |\mathbf{v}|_k$.

The following theorem is a first piece of evidence that this definition is useful.

Theorem 1.4. *Suppose there exists a box metric Φ and an $L > 1$ for which Γ is box expansive. Then f is hyperbolic.*

In particular, there exists a smooth function $\rho: \mathcal{B}(\Gamma) \rightarrow \mathbb{R}^+$, which defines a riemannian metric on $T\mathcal{B}$ (by ρ times euclidean), such that for all $z \in \mathcal{B}$, and $\mathbf{v} \in T_z \mathcal{B}$, we have $|D_z f(\mathbf{v})|_\rho \geq L |\mathbf{v}|_\rho$.

The key step in the proof of this theorem is to smooth out the box metric Φ in a small neighborhood of the box boundaries, using a partition of unity argument and the “edge overlap” factor δ from Definition 1.1. The proof is given in Section 2.

We develop an algorithm for attempting to build a box metric for which a given Γ is box expansive, called the *Handicap Hedging Algorithm*, in Section 4. Then we describe how any outcome of the algorithm gives useful dynamical information, in Section 5. For example, we obtain:

Theorem 1.5. *Given Γ and $L > 0$, the Handicap Hedging Algorithm either*

- (1) *constructs a box metric for which Γ is box expansive by L , or*
- (2) *produces a cycle of Γ which is an obstruction, showing there exists no box metric for which Γ is box expansive by L .*

Note that showing box expansion by $L = 1$ may be instructive, but it is not enough to prove hyperbolicity. Thus, if any algorithm, for example, the Handicap Hedging Algorithm, builds a box metric for which some Γ is box expansive by some $L > 1$, then f is proven hyperbolic.

1.2. Secondary results and discussion of the approach. A natural question one might ask is whether there is a simple algorithm for showing box expansion, without constructing a box metric; for example, by examine the multipliers along the simple cycles in the graph. (A cycle in a graph is called *simple* if it is composed of distinct vertices.) In fact, examining cycle multipliers is the key idea to a better understanding of box expansion and box metrics. We would like to thank Clark Robinson for asking this question.

Definition 1.6. Call $\lambda_k = \min\{|f'(z)| : z \in B_k\}$ the *multiplier* of B_k . If $B_0 \rightarrow \dots \rightarrow B_{n-1} \rightarrow B_n = B_0$ is an n -cycle of boxes in Γ , then the *cycle multiplier* is $(\lambda_0 \cdots \lambda_{n-1})$, and the *average cycle multiplier* is $(\lambda_0 \cdots \lambda_{n-1})^{1/n}$.

Along the way to proving Theorem 1.5, we establish the following characterizations of box expansion, which are independent of any algorithm used to find an expanded box metric.

Proposition 1.7. *Let \mathcal{L} be the minimum average cycle multiplier over all simple cycles in the graph Γ . Then for any $L > 0$, there exists a box metric for which Γ is box expansive by L if and only if $L \leq \mathcal{L}$.*

Corollary 1.8. *Let \mathcal{M} be the minimum cycle multiplier over all simple cycles in the graph Γ . Then Γ is box expansive (hence f is hyperbolic) if and only if $\mathcal{M} > 1$.*

Thus if some Γ is box expansive, then each cycle multiplier for Γ is greater than one. However, some boxes in a cycle could have multiplier less than one, if others are large enough to compensate. In this case, in the euclidean metric, the map is not expansive along every edge. The handicaps are designed to spread the expansion out along the cycles, so that in the box metric, the map is expansive by at least L on every edge in the graph. This is why we use the term “handicap”.

According to Corollary 1.8, we can show whether f is box expansive simply by computing \mathcal{M} . There do exist efficient algorithms for finding such a minimum, see [4]. However, in order to have explicit information about the hyperbolic structure, we still want to find a viable box expansion constant $L < \mathcal{L}$, and build an expanded box metric, which we can do using the Handicap Hedging Algorithm.

One weakness of this algorithm is that the box expansion constant, $L > 1$, must be inputted in advance. Proposition 1.7 shows the ideal box expansion constant is the smallest average cycle multiplier, \mathcal{L} . However, in Section 6.1, we describe counterexamples which suggest that the only algorithm for explicitly computing \mathcal{L} is exponential (thus too inefficient for our examples). But then we describe an efficient method for finding a good approximation to the ideal \mathcal{L} , in Section 6.2.

Our approach for testing hyperbolicity has some similarities to work of Osipenko ([17, 18]). For f a diffeomorphism of a compact Riemannian manifold $M \subset \mathbb{R}^n$, he uses a symbolic image Γ of f , and develops a general algorithm for describing the expansion of f by approximating Lyapunov exponents and the Morse spectrum of the chain recurrent set. His algorithm can also be used to verify hyperbolicity. Our work here differs in that we are interested in developing and implementing efficient algorithms for families of polynomial maps of \mathbb{C} (of degree greater than one). For this study, we found it more efficient to get expansion information and a hyperbolicity test by constructing a metric expanded by f . Osipenko’s hyperbolicity test has the same computational complexity as finding the smallest average cycle multiplier for all simple cycles in the graph Γ (see Section 6.1).

In implementation, we control round-off error using *interval arithmetic* (IA). This method was recommended by Warwick Tucker, who used it in his recent computer proof that the Lorenz differential equation has the conjectured geometry ([22]). In designing our algorithms, we must keep in mind the workings of IA. We thus give a brief description of IA in Section 3.

To summarize the organization of the remaining sections: in Section 2 we show box expansion implies the standard definition of expansion, to prove Theorem 1.4; in Section 3, we briefly describe interval arithmetic; in Section 4 we give our basic algorithm, the Handicap Hedging Algorithm, for attempting to establish box expansion; in Section 5 we obtain dynamical information from either success or failure of the Handicap Hedging Algorithm, proving Theorem 1.5, Proposition 1.7, and Corollary 1.8; in Section 6 we compare and contrast ideal versus efficient methods for determining a good expansion constant L ; and in Section 7 we discuss our implementation of the algorithm and give examples of output.

2. BOX EXPANSION IMPLIES CONTINUOUS EXPANSION

In this section, we show box expansion implies the standard definition of expansion. Throughout, let f denote a polynomial map of \mathbb{C} of degree $d > 1$, and as in Definition 1.1, let Γ be a box chain model of f on J , with vertex set $\mathcal{V} = \{B_k\}_{k=1}^N$, composing the box Julia set $\mathcal{B} = \cup_{k=1}^N B_k \supset J$.

First, it is more natural for computer calculations, and reduces round-off error, to consider vectors in \mathbb{R}^2 , rather than \mathbb{C} , and use the L^∞ metric of \mathbb{R}^2 , rather than euclidean. Hence, we consider

$$|z| = \max\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\}.$$

Also, let $\mathcal{N}(S, r)$ denote the open r -neighborhood about the set S in the metric induced by the above. This metric is *uniformly equivalent* to the euclidean metric $|\cdot|_e$, since $\frac{1}{\sqrt{2}}|\cdot|_e \leq |\cdot| \leq |\cdot|_e$. Thus neighborhoods are slightly different, but the topology generated by them is exactly the same, so they can nearly be used interchangeably.

To prove Theorem 1.4, we use a partition of unity to smooth out a box metric Φ . First, we need a lemma from [9].

Lemma 2.1 ([9]). *There exists an $\eta > 0$ so that if $B_k, B_j \in \mathcal{V}$, with $z \in \mathcal{N}(B_k, \eta)$ and $f(z) \in \mathcal{N}(B_j, \eta)$, then there is an edge from B_k to B_j in Γ .*

To prove this lemma, we used the assumption that f was a polynomial of degree $d > 1$, and the fact that by Definition 1.1, there is a $\delta > 0$ such that there is an edge from B_k to B_j if a δ -neighborhood of $f(B_k)$ intersects B_j . Now, we can obtain:

Lemma 2.2. *Suppose Γ is box expansive. Then there exists a $\tau > 0$ such that if $B_k, B_j \in \mathcal{V}$, and $z \in \mathcal{B} \cap \mathcal{N}(B_k, \tau)$ with $f(z) \in \mathcal{N}(B_j, \tau)$, then for any $\mathbf{v} \in T_z \mathbb{C}$, $|D_z f(\mathbf{v})|_j \geq L|\mathbf{v}|_k$.*

Proof. Among other requirements below, let $\tau > 0$ be less than η from Lemma 2.1. Then for z satisfying the hypotheses, there is an edge from B_k to B_j in Γ .

Note since we are working in one dimension, $D_x f = f'(x)$, hence box expansion yields that for $x \in B_k$, $\varphi_j |f'(x)| |\mathbf{v}| \geq L\varphi_k |\mathbf{v}|$, thus simply $\varphi_j |f'(x)| \geq L\varphi_k$.

Since \mathcal{B} is compact, \mathcal{V} is finite, and $f'(x)$ is continuous, there is an $\alpha \geq 0$ so that:

- (1) $\alpha = \min\{\varphi_j |f'(x)| - L\varphi_k : x \in B_k, (k, j) \in \mathcal{E}\},$
- (2) if $\tau < \eta$ is sufficiently small, then for any j , $|x - z| < \tau$ implies that $\varphi_j |f'(x) - f'(z)| < \alpha$.

Now z is not necessarily in B_k , but $z \in \mathcal{B}$, so suppose $z \in B_m$ and $x \in B_m \cap B_k$ such that $|x - z| < \tau$. Then $\varphi_j |f'(x) - f'(z)| < \alpha$; further, there is an edge $(k, j) \in \mathcal{E}$, hence x satisfies $\varphi_j |f'(x)| - L\varphi_k \geq \alpha$. Combining these gives $\varphi_j |f'(z)| \geq L\varphi_k$. Thus $|D_z f(\mathbf{v})|_j \geq L|\mathbf{v}|_k$. \square

Now we use the τ -overlap to convert a box metric into a riemannian metric.

Definition 2.3. Suppose Γ is box expansive for a box metric Φ . Let $\tau > 0$ be as given by Lemma 2.2. Define a partition of unity on $\mathcal{B}(\Gamma)$ by choosing smooth functions $\rho_k : \mathbb{C} \rightarrow [0, 1]$, for each box $B_k \in \mathcal{V}$, such that $\operatorname{supp}(\rho_k) \subset \mathcal{N}(B_k, \tau)$ and $\sum_k \rho_k(x) = 1$, for any $x \in \mathcal{B}$. Define the smooth function $\rho = \rho(\Phi) : \mathcal{B} \rightarrow [0, 1]$ by $\rho(x) = \sum_k \rho_k(x) \varphi_k$. Then ρ induces a riemannian metric on $T\mathcal{B}$, with a smoothly

varying norm, $|\cdot|_\rho$, where if $x \in \mathcal{B}$ and $\mathbf{v} \in T_x\mathcal{B}$ then

$$|\mathbf{v}|_\rho = |\mathbf{v}| \rho(x) = |\mathbf{v}| \sum_k \rho_k(x) \varphi_k = \sum_k \rho_k(x) |\mathbf{v}|_k.$$

Note $|\cdot|_\rho$ and $|\cdot|_\Phi$ are very close. They only differ in the small τ -neighborhoods of the box boundaries, where the ρ metric smooths out the Φ metric.

It is also straightforward to show that both the ρ metric and the Φ metric are uniformly equivalent to euclidean, with

$$\left(\min_k \{\varphi_k\} \right) |\cdot| \leq |\cdot|_\rho, |\cdot|_\Phi \leq \left(\max_k \{\varphi_k\} \right) |\cdot|.$$

We establish Theorem 1.4 by showing that if Γ is box expansive for a box metric Φ , then f is expansive in \mathcal{B} for the metric $\rho(\Phi)$.

Proof of Theorem 1.4. Let $|\cdot|_\rho, \tau$ be as in Definition 2.3.

Thus τ is small enough that if $x \in \text{supp}(\rho_k)$, then for any j such that $f(x) \in \text{supp}(\rho_j)$, we have $\varphi_j |D_x f(\mathbf{v})| \geq L \varphi_k |\mathbf{v}|$, for any $\mathbf{v} \in T_x\mathbb{C}$. Then if we set

$$\varphi_x = \max\{\varphi_k : x \in \text{supp}(\rho_k)\}, \quad \text{and} \quad \varphi_{f,x} = \min\{\varphi_j : f(x) \in \text{supp}(\rho_j)\},$$

we know $\varphi_{f,x} |D_x f(\mathbf{v})| \geq L \varphi_x |\mathbf{v}|$, for any $\mathbf{v} \in T_x\mathbb{C}$.

Now if we use that $\sum_k \rho_k(x) = \sum_j \rho_j(f(x)) = 1$, we get the result:

$$\begin{aligned} |D_x f(\mathbf{v})|_\rho &= |D_x f(\mathbf{v})| \sum_j \rho_j(f(x)) \varphi_j \geq |D_x f(\mathbf{v})| \varphi_{f,x} \\ &\geq L \varphi_x |\mathbf{v}| \geq L |\mathbf{v}| \sum_k \rho_k(x) \varphi_k = L |\mathbf{v}|_\rho. \end{aligned}$$

□

Thus if any box model Γ of f on J is box expansive by some $L > 1$, then f is expanding on J in the riemannian metric ρ , and hence f is hyperbolic.

3. INTERVAL ARITHMETIC

Interval arithmetic (IA) provides a natural and efficient method for manipulating boxes, and also for maintaining rigor in computations. The basic objects of IA are closed intervals, $[a] = [\underline{a}, \bar{a}] \in \mathbb{IK}$, with end points in some fixed field, \mathbb{K} . An arithmetic operation on two intervals produces a resulting interval which contains the real answer. For example,

$$\begin{aligned} [a] + [b] &:= [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \\ [a] - [b] &:= [\underline{a} - \bar{b}, \bar{a} - \underline{b}] \end{aligned}$$

Multiplication and division can also be defined in IA.

Computer arithmetic is performed not with real numbers, but rather in the finite space \mathbb{F} of numbers representable by binary floating point numbers of a certain finite length. For example, since the number 0.1 is not a dyadic rational, it has an infinite binary expansion, so is not in \mathbb{F} .

Since an arithmetical operation on two numbers in \mathbb{F} may not have a result in \mathbb{F} , in order to implement rigorous IA we must round outward the result of any interval arithmetic operation, *e.g.* for $[a], [b] \in \mathbb{IF}$,

$$[a] + [b] := [\downarrow \underline{a} + \underline{b}, \uparrow \bar{a} + \bar{b}],$$

where $\lfloor x \rfloor$ denotes the largest number in \mathbb{F} that is strictly less than x (i.e., x rounded down), and $\lceil x \rceil$ denotes the smallest number in \mathbb{F} that is strictly greater than x (i.e., x rounded up). This is called IA with *directed rounding*.

For any $x \in \mathbb{R}$, let $\text{Hull}(x)$ be the smallest interval in \mathbb{F} which contains x . That is, if $x \in \mathbb{F}$, then $\text{Hull}(x)$ denotes $[x, x]$. If $x \in \mathbb{R} \setminus \mathbb{F}$, then $\text{Hull}(x)$ denotes $[\lfloor x \rfloor, \lceil x \rceil]$.

In higher dimensions, IA operations can be carried out component-wise, on *interval vectors*. Note a box in $\mathbb{C} = \mathbb{R}^2$ is simply an interval vector.

In designing our algorithms, every arithmetical calculation is carried out with IA, and thus we must think carefully about how to use IA in each situation. Our extensive use of boxes is designed to make IA calculations natural. However, IA tends to create problems with propagating increasingly large error bounds, if not handled carefully. For example, iterating a polynomial map on an interval which is close to the Julia set, J , can produce a tremendously large interval after only a few iterates (due to the expanding behavior of the map near J). Thus, in the remainder of the paper, after describing each algorithm we note how IA is being used.² The interested reader can find an abundance of materials on IA, for example [3, 13, 14].

4. THE HANDICAP HEDGING ALGORITHM

In this section we describe our basic algorithm, the Handicap Hedging Algorithm, for attempting to build an expanded box metric for a box chain model Γ .

The problem of finding a set of handicaps $\{\varphi_k\}$ defining a box metric Φ for which Γ is box expansive by a given $L > 1$ is strictly a graph theoretic problem. We want to find handicaps so that for every edge $(k, j) \in \mathcal{E}$, for every point $z \in B_k$, and for every vector $\mathbf{v} \in T_z B_k$, we have

$$\varphi_j |D_z f(\mathbf{v})| = \varphi_j |D_z f| |\mathbf{v}| \geq L \varphi_k |\mathbf{v}|,$$

or equivalently, if $\lambda_k = \min_{z \in B_k} |f'(z)|$ is the multiplier of B_k , we want

$$(1) \quad \varphi_j \geq L \varphi_k / \lambda_k.$$

For polynomial maps of \mathbb{C} , a lower bound for the multiplier λ_k in a box is easily calculated with interval arithmetic. After the λ_k are calculated, we can forget the map f and simply think of Γ as a strongly connected directed graph endowed with edge weights, $\xi_{k,j} = L/\lambda_k$, if $(k, j) \in \mathcal{E}$. In the case of maps in one dimension, the edge weights are all equal along all edges emanating from a single vertex, i.e., $\xi_{k,1} = \xi_{k,2} = \dots \xi_{k,n}$. However we describe the algorithm for the more general situation, where $\xi_{k,j} \neq \xi_{k,m}$, since it is not more difficult, and it is useful for higher dimensional applications (for example, in [8]).

Definition 4.1. Let Γ be a directed graph with positive edge weights $\Xi = \{\xi_{k,j} : (k, j) \in \mathcal{E}\}$. Let $\Phi = \{\varphi_k : v_k \in \mathcal{V}\}$ be a set of positive vertex weights, called *handicaps*. We call Φ *consistent handicaps* (for Γ, Ξ) if $\varphi_j \geq \xi_{k,j} \varphi_k$, for every edge $(k, j) \in \mathcal{E}$.

If equality holds on every edge, then we call Φ *strict handicaps*.

Then Γ is box expansive if there exists a set of consistent handicaps Φ for Γ , given the edge weights suggested by (1).

To attempt to find consistent handicaps given Γ and the edge weights Ξ , we break the problem into a finite induction. Step 0 consists of finding an initial set

²All of our IA computations use the PROFIL/BIAS package, available at [20].

of strict handicaps for some spanning tree, Γ_0 , of Γ . At steps $n > 0$, we choose a graph Γ_n such that $\Gamma_0 \subset \Gamma_n \subset \Gamma$, then seek a consistent set of handicaps for Γ_n .

Definition 4.2. A directed graph Γ_0 is an *arborescence* if there is a root vertex v_0 so that for any other vertex v , there is a unique simple path from v_0 to v . Such a graph is a tree, and must have exactly one incoming edge for each vertex $v \neq v_0$.

If Γ is strongly connected, then for each vertex v_0 in Γ , there is a minimum spanning tree Γ_0 with root vertex v_0 which is an arborescence (simply perform a depth first or breadth first search from v_0). We call such a Γ_0 a *spanning arborescence*.

Definition 4.3. Let Γ be a finite, strongly connected directed graph. Let $\Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_N = \Gamma$ be a nested sequence of subgraphs of Γ such that Γ_0 is a spanning arborescence of Γ , and for $n \geq 1$, $\mathcal{V}(\Gamma_n) = \mathcal{V}(\Gamma)$, and $\mathcal{E}(\Gamma_n)$ is formed by adding one edge of $\mathcal{E}(\Gamma) \setminus \mathcal{E}(\Gamma_{n-1})$ to $\mathcal{E}(\Gamma_{n-1})$. We call such a sequence an *edge exhaustion* of Γ .

Note by the above definitions that an edge exhaustion exists for any finite, strongly connected directed graph Γ . Note also that since Γ_0 is a spanning arborescence, each Γ_n is edge connected.

Algorithm 4.4 (Recursively hedging handicaps via an edge exhaustion). Let $\Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_N = \Gamma$ be an edge exhaustion of Γ .

Base Case: Inductively construct a set of *strict* handicaps, $\Phi^0 = \{\varphi_k^0\}$ for Γ_0 , by choosing any φ_0^0 for the root vertex v_0 , then pushing this value across Γ_0 by multiplication with successive edge weights. That is, if $v_k \in \mathcal{V}(\Gamma_0)$, and $v \neq v_0$, then since Γ_0 is an arborescence, the predecessor, $v_{\pi(k)}$, is uniquely defined such that the edge $(\pi(k), k)$ is in Γ_0 . Then set $\varphi_k^0 = \xi_{k, \pi(k)} \varphi_{\pi(k)}^0$.

For our box expansion application, we pick a random vertex as the root, and start with $\varphi_0^0 = 1$, *i.e.*, take the euclidean metric on that box.

Inductive Step: Suppose we have a set of consistent handicaps $\Phi^{n-1} = \{\varphi_k^{n-1}\}$ for Γ_{n-1} . We attempt to define a set of consistent handicaps $\Phi^n = \{\varphi_k^n\}$ for Γ_n , by adjusting the set Φ^{n-1} , using a process we call *recursively hedging the handicaps*. First, start with the set of temporary handicaps $\Phi^n := \Phi^{n-1}$ on Γ_n . Let (v_a, v_b) denote the edge in Γ_n that is not in Γ_{n-1} (this edge is unique by definition of edge exhaustion). Then we adjust the b th handicap, by doing

$$\begin{aligned} &\text{if } \varphi_b^n < \xi_{a,b} \varphi_a^n \\ &\quad \text{set } \varphi_b^n = \xi_{a,b} \varphi_a^n \end{aligned}$$

If the second line above is performed, *i.e.* φ_b^n is strictly increased, then we say we have *hedged* φ_b^n *along the edge* (v_a, v_b) . But now if φ_b^n is hedged, then in order to keep consistency of the handicaps, for each vertex v_c in Γ_n which is adjacent to v_b , we may need to hedge φ_c^n along the edge (v_b, v_c) :

$$\begin{aligned} &\text{set } \varphi_b^n = \xi_{a,b} \varphi_a^n \\ &\text{for each } v_c \text{ in } \Gamma_n \text{ and in the adjacency list of } v_b \text{ do} \\ &\quad \text{if } \varphi_c^n < \xi_{b,c} \varphi_b^n \\ &\quad \quad \text{set } \varphi_c^n = \xi_{b,c} \varphi_b^n \end{aligned}$$

The hedging process becomes recursive here, for if any φ_c^n is increased, then we may need to hedge handicaps along each edge in Γ_n emanating from v_c , etc. The worst case scenario is that every vertex reachable from v_b along every path in Γ_n would need to be hedged.

However if at any step in the hedging, we find a vertex v_x whose handicap need not be increased, *i.e.*, $\varphi_x^n \geq \xi_{\pi(x),x} \varphi_{\pi(x)}^n$ already, then we may stop, since the handicaps of vertices reachable from v_x will not need to be increased. This greatly saves on computational time. Thus, our process so far is:

```

Hedge( $a, b$ ) =
  if  $\varphi_b^n < \xi_{a,b} \varphi_a^n$ 
    set  $\varphi_b^n = \xi_{a,b} \varphi_a^n$ 
    for each  $v_c$  in  $\Gamma_n$  and in the adjacency list of  $v_b$  do
      Hedge( $b, c$ )

```

Will this recursive procedure always terminate in a consistent set of handicaps Φ^n for Γ_n ? Of course not, since it is possible that there does not exist any set of consistent handicaps for Γ with edge weights Ξ . There is one possible obstruction: suppose that for the new edge in Γ_n , edge (v_a, v_b) , we travel away from v_b , hedging the handicaps along every edge in some path, $v_a, v_b, v_c, \dots, v_x$, and then discover v_a in the adjacency list of v_x . This means every handicap on a path in Γ_n from v_b to v_x has been increased, since we stopped the search if it was not. The handicap at v_a cannot be increased, since then we would have to hedge φ_b^n along (v_a, v_b) again, and an infinite loop of hedging along this path through v_x would occur.

Thus the cycle $\{v_a, v_b, v_c, \dots, v_x, v_a\}$ could be an obstruction to consistent handicaps, for upon seeing v_a from v_x , we can only check if $\varphi_a^n \geq \xi_{x,a} \varphi_x^n$ already. This leads us to the following algorithm for the n^{th} inductive step:

```

do Hedge( $a, b, a$ ), where
Hedge( $u, v, a$ ) =
  if  $\varphi_v^n \geq \xi_{u,v} \varphi_u^n$ , then return 1
  else if ( $v = a$ ), then return 0
  else
    set  $\varphi_v^n = \xi_{u,v} \varphi_u^n$ 
    for each  $w$  in  $\Gamma_n$  and in adjacency list of  $v$  do
      Hedge( $v, w, a$ )

```

Thus the n^{th} step is terminated when either a cyclic obstruction is found (at the “return 0” line above), or when all necessary hedgings have successfully been performed (if **Hedge**(a, b, a) returns 1). If no cyclic obstructions are found, then Φ^n is a consistent set of handicaps.

Thus Algorithm 4.4 either terminates in a cyclic obstruction at some Γ_n , or produces consistent handicaps for all of $\Gamma = \Gamma_N$. This dichotomy leads to Theorem 1.5, proved in Section 5.

The Handicap Hedging Algorithm consists simply of applying Algorithm 4.4 to the case of showing box expansion for a box chain model Γ of f on J . We describe this process more explicitly with the following pseduo-code. Comments are parenthetical and to the right. The routine consists of a main function, **BuildMetric**, and its two helper sub-routines, **SpanTree** and **Hedge**. **SpanTree** creates an arborescence using a breadth first search and a queue (first-in = first-out) to traverse the graph. **Hedge** uses a depth first style to check the rest of the edges. See [4] for background on graph searches, and note the remark following the pseudo-code on why these styles were chosen.

Algorithm 4.5 (The Handicap Hedging Algorithm).

BuildMetric(Γ, L):

```

for every vertex  $u$  in  $\Gamma$ 
  set  $\text{color}[u]$  to white
  for each vertex  $v$  in adjacency list of  $u$  do
    set  $\text{check}[u][v] = 0$ 
set  $\varphi_0 = 1$ 
do SpanTree(vertex 0 of  $\Gamma$ )
for every edge  $(a, b)$  in  $\Gamma$ 
  if ( $\text{check}[a][b] = 0$ ) then      (edge  $(a, b)$  not in previous step of exhaustion)
    set  $\text{check}[a][b] = 1$           (put  $(a, b)$  into current step of exhaustion)
    if (Hedge( $a, b, a, L$ ) = 0) then return 0
return 1                          (if get this far, then successful)

```

SpanTree(u):

```

put vertex  $u$  on the queue,  $Q$  (at the end)
set  $\text{color}[u]$  to gray
while  $Q \neq \emptyset$ 
  let  $v$  be the head (first element) of the queue
  for each vertex  $w$  in adjacency list of  $v$  do
    if  $\text{color}[w]$  is white
      put vertex  $w$  on the queue,  $Q$  (at the end)
      set  $\text{color}[w]$  to gray
      set  $\text{check}[v][w] = 1$       (record edge  $(v, w)$  is in spanning tree)
      set  $\varphi_w = L\varphi_v/\lambda_v$ 
  remove first element from queue    (remove  $v$ )
  set  $\text{color}[v]$  to black

```

Hedge(u, v, a, L):

```

if ( $\varphi_v \geq L\varphi_u/\lambda_u$ ) then return 1      (i.e., edge  $(u, v)$  is already ok)
else if ( $v = a$ ) then return 0              (we cannot increase  $\varphi_a$ , so fails)
else
  set  $\varphi_v = L\varphi_u/\lambda_u$                   (increase  $\varphi_v$ )
  for each  $w$  in adjacency list of  $v$  do
    if ( $\text{check}[v][w] = 1$ ) then      ( $(v, w)$  is in current step of exhaustion)
      if (Hedge( $v, w, a, L$ ) = 0) then return 0
return 1                                    (if get this far, then successful)

```

Remark 2. In **SpanTree** we maximize efficiency by using a breadth first style search to traverse the graph, instead of depth first. Running depth first search on a typical box chain model Γ tends to produce a spanning arborescence with a very long path, whereas breadth first search constructs an arborescence with paths of minimum length. This is because for a polynomial map f , the typical Γ has a large number of vertices and a small bound on the out-degree of vertices (related to the bound on the derivative of the map in \mathcal{B}). In addition, Γ is strongly connected.

Long paths creates two problems. First, creating such a tree can cause a memory overflow, due to many nested recursive function calls. Additionally, even when there is no crash, a large path tends to create an initial metric farther from Euclidean. See Example 6.1 in Section 6 for an exploration of the latter phenomena.

Recall in our implementation, we control rounding using interval arithmetic (Section 3). In the above, we round up to ensure the inequality is satisfied, *i.e.*, check $\varphi_v \geq \sup(\text{Hull}(L) * \text{Hull}(\varphi_u) / \text{Hull}(\lambda_u))$.

5. CHARACTERIZATION OF BOX EXPANSION

Recall from Definition 1.6 that the multiplier of a box B_k is $\lambda_k = \min_{z \in B_k} |f'(z)|$, and for an n -cycle of boxes $(B_0 \rightarrow \dots \rightarrow B_{n-1} \rightarrow B_n = B_0)$ in Γ , we call $(\lambda_0 \cdots \lambda_{n-1})$ the cycle multiplier and $(\lambda_0 \cdots \lambda_{n-1})^{1/n}$ the average cycle multiplier. Also, a cycle in a graph is called *simple* if it is composed of distinct vertices.

We can now specify the implications of success or failure in the Handicap Hedging Algorithm, proving Theorem 1.5, Proposition 1.7, and Corollary 1.8.

Proposition 5.1. *For $L > 0$, the Handicap Hedging Algorithm, *i.e.*, Algorithm 4.5, or Algorithm 4.4 with edge weights $\Xi = \{\xi_{k,j} = L/\lambda_k : (k,j) \in \mathcal{E}\}$, either constructs a set of consistent handicaps showing Γ is box expansive by L , or finds an n -cycle of boxes with cycle multiplier less than L^n .*

Proof. We observed in the description of Algorithm 4.4 that the only obstruction to building consistent handicaps is if in adding an edge (u,v) , we find a cycle of boxes $(u = B_0 \rightarrow v = B_1 \rightarrow \dots \rightarrow B_{n-1} \rightarrow B_n = B_0)$, such that, holding φ_0 fixed, in order to keep consistency the metric handicaps must be increased along every edge in the cycle. That is, $\varphi_{k+1} = L\varphi_k/\lambda_k$ for $0 \leq k \leq n-2$, and we have the failure $\varphi_0 < L\varphi_{n-1}/\lambda_{n-1}$. But then

$$\varphi_0 < \frac{L\varphi_{n-1}}{\lambda_{n-1}} = \frac{L^2\varphi_{n-2}}{\lambda_{n-1}\lambda_{n-2}} = \dots = \frac{L^n\varphi_0}{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}.$$

Hence, $\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0 < L^n$. \square

Lemma 5.2. *Let $(B_0 \rightarrow \dots \rightarrow B_{n-1} \rightarrow B_n = B_0)$ be an n -cycle of boxes in Γ , such that consistent handicaps $\{\varphi_0, \dots, \varphi_{n-1}, \varphi_n = \varphi_0\}$ can be chosen to show f box expands by $L > 1$ along the cycle. Then its cycle multiplier is at least L^n .*

Proof. By hypothesis, we know $\varphi_{k+1}\lambda_k \geq L\varphi_k$, $k \in \{0, \dots, n-1\}$. Thus,

$$\lambda_0 \cdots \lambda_{n-1} \geq L \frac{\varphi_0}{\varphi_1} L \frac{\varphi_1}{\varphi_2} \cdots L \frac{\varphi_{n-1}}{\varphi_n = \varphi_0}.$$

Cross cancellation and simplifying leaves only $\lambda_0 \cdots \lambda_{n-1} \geq L^n$. \square

Lemma 5.3. *For $L > 0$, there is a cycle of boxes in Γ with average multiplier less than L if and only if there is no box metric which Γ expands by L .*

Proof. First, Lemma 5.2 gives the forward implication immediately; for, if there is a box metric which Γ expands by L , then every cycle must have average multiplier at least L . Conversely, suppose there is no box metric which Γ expands by L . Then by Proposition 5.1, there exists a cycle with average multiplier less than L . \square

Proof of Theorem 1.5. Proposition 5.1 shows that given $L > 0$, the Handicap Hedging Algorithm either constructs a box metric for which Γ is expansive by L , or finds a cycle with average multiplier less than L . Lemma 5.3 shows the latter is equivalent to saying there is no box metric which Γ expands by L . \square

Proof of Proposition 1.7. This follows directly from Lemma 5.3. Indeed, let $L > 0$ be given. First, assume $L \leq \mathcal{L}$. Then all cycles have average multiplier at least L , so by Lemma 5.3 there exists a box metric which Γ expands by L . Next, assume $L > \mathcal{L}$. Then the cycle of boxes with average multiplier \mathcal{L} has average multiplier less than L , hence by Lemma 5.3 there is no box metric which Γ expands by L . \square

Proof of Corollary 1.8. This follows easily from Proposition 1.7, using the fact that $a > 1$ if and only if $a^{1/m} > 1$, for any positive a and m . Let \mathcal{L} denote the smallest average cycle multiplier over all simple cycles in the graph. By Proposition 1.7, Γ is box expansive if and only if $\mathcal{L} \geq 1$. Let m be the length of the cycle with multiplier \mathcal{M} . Let n be the length of the cycle with average multiplier \mathcal{L} . Then $\mathcal{M}^{1/m} \geq \mathcal{L}$ and $\mathcal{M} \leq \mathcal{L}^n$. Suppose Γ is box expansive. Then $\mathcal{M}^{1/m} \geq \mathcal{L} > 1$, hence $\mathcal{M} > 1$. Conversely, suppose $\mathcal{M} > 1$. Then $\mathcal{L}^n \geq \mathcal{M} > 1$, hence $\mathcal{L} > 1$. Thus Γ is box expansive. \square

6. FINDING A GOOD BOX EXPANSION CONSTANT L

In theory and in practice, a box expansion constant more appropriate for a particular collection of boxes yields a “better” metric. In particular, in running the program Hypatia it is easy to find behavior like the following. Suppose for some map, we successfully build a metric with box expansion constant $L = 1.2$, but the resulting handicaps range between 4.3×10^{-18} and 0.6. Depending on the map f , trying $L = 1.5$ could improve the handicaps to be in the range 0.068 to 0.55. This yields a metric more computationally tractable, and closer to euclidean. Why do we see this behavior? Consider the example:

Example 6.1. Suppose there is a path of boxes $B_0 \rightarrow \dots \rightarrow B_n$ such that (for simplicity) all multipliers are approximately the same value λ . Then suppose we try to show box expansion by some $L < \lambda$. We would define the handicaps in the boxes of this cycle so that $\varphi_{k+1} \geq L\varphi_k/\lambda$, for $0 \leq k \leq n-1$. Thus we would get

$$\varphi_1 = \frac{\varphi_0 L}{\lambda}, \varphi_2 = \frac{\varphi_1 L}{\lambda} = \frac{\varphi_0 L^2}{\lambda^2}, \dots, \varphi_n = \frac{\varphi_0 L^n}{\lambda^n}.$$

But since $L < \lambda$, large n leads to $L^n \ll \lambda^n$ and $\varphi_n \ll \varphi_0$. Thus, if we use an L which is “too small”, then the handicaps plummet.

In fact in running Hypatia, we have found that it is easy for a bad choice of L to lead to handicaps which are so large or small that the machine cannot distinguish them from 0 or ∞ . Checks must be put in place in the algorithms to flag such occurrences. This can be guarded against somewhat by using breadth first search style algorithms as much as possible, rather than depth first search. In particular, in **SpanTree**, if a depth first search is used, then the spanning arborescences usually contain very long paths. On longer paths, as Example 6.1 illustrates, the handicaps are more likely to get unmanageable.

6.1. An optimal, yet impractical solution. Proposition 1.7 implies that the ideal box expansion constant, \mathcal{L} , is equal to the smallest average cycle multiplier over all simple cycles. This can *theoretically* be computed, since the graph is finite. However, in this section we shall see the computation seems unwieldy.

For a first approach to finding \mathcal{L} , recall that there do exist efficient algorithms for finding the cycle with the smallest cycle multiplier ([4]). Unfortunately, the

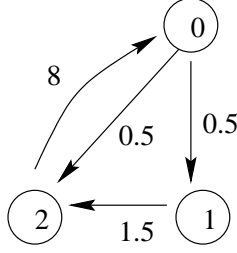


FIGURE 1. Graph illustrating Examples 6.2 and 6.4

following example shows that the smallest average cycle multiplier cannot be simply derived from examining the cycle with the smallest cycle multiplier.

Example 6.2. Consider the graph with three vertices $\{0, 1, 2\}$, and with the four edges $\{(0, 1), (1, 2), (0, 2), (2, 0)\}$, shown in Figure 1. It has two cycles: $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ and $0 \rightarrow 2 \rightarrow 0$, with multipliers: $\lambda_0 = 1/2, \lambda_1 = 3/2, \lambda_2 = 8$. The cycle multipliers are: $(\lambda_0 \lambda_2) = 4$ and $(\lambda_0 \lambda_1 \lambda_2) = 6$. Thus the smallest cycle multiplier is along the cycle $0 \rightarrow 2 \rightarrow 0$. The average along this cycle is $4^{1/2} = 2$. However, the average cycle multiplier along the other cycle is $(\lambda_0 \lambda_1 \lambda_2)^{1/3} = 6^{1/3} = 1.82$.

Thus, we need an algorithm for computing the smallest average cycle multiplier from scratch. For our second approach to this computation, we might posit that we could adapt the algorithm used to compute the smallest cycle multiplier, to instead compute the smallest average cycle multiplier. This algorithm involves considering “best paths” during a breadth first search of the graph. However, in the following we explain how the “average” prevents this shortcut from working.

Definition 6.3. If $P = (B_0 \rightarrow \dots \rightarrow B_n)$ is a path in the graph Γ , then the *path multiplier* is the product of the multipliers along the path, $\lambda(P) := \lambda_0 \cdots \lambda_{n-1}$, and the *average path multiplier* is $(\lambda(P))^{1/n}$.

Let B_k and B_j be vertices in a graph Γ . Let $P_{k,j}$ be the path from B_k to B_j with the smallest average path multiplier of all paths from B_k to B_j . The following example shows that there exist graphs Γ such that the cycle containing B_k and B_j with the smallest average multiplier does not contain the path $P_{k,j}$.

Example 6.4. Again consider the three vertex graph in Figure 1, used in the proof of Example 6.2. It has two cycles: $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ and $0 \rightarrow 2 \rightarrow 0$, and has multipliers: $\lambda_0 = 1/2, \lambda_1 = 3/2, \lambda_2 = 8$. Comparing paths from vertex 0 to vertex 2, we see $\lambda_0 = 1/2$ and $(\lambda_0 \lambda_1)^{1/2} = \sqrt{3}/2$. The smallest average path multiplier is along the path $0 \rightarrow 2$. However, Example 6.2 shows the smallest average cycle multiplier is along the cycle $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$.

Example 6.4 suggests that an algorithm to compute the smallest average cycle multiplier in a graph must compute the average cycle multiplier for each simple cycle separately. A simple combinatorial argument shows that the number of simple cycles in a graph (even a sparse graph) is exponential in the number of vertices of the graph. The graphs created for polynomial maps of \mathbb{C} are large enough to make an exponential algorithm prohibitively inefficient (see Section 7 for several examples of such graphs). Thus, we cannot expect an efficient algorithm for determining the ideal box expansion constant.

The previous exploration can also be used to examine the computational complexity of Osipenko's method ([17, 18]) for approximating Lyapunov exponents, or the Morse spectrum, on J , and the subsequent hyperbolicity test. If $C = B_0 \rightarrow \dots \rightarrow B_{n-1} \rightarrow B_n = B_0$ is an n -cycle of boxes in Γ , then the characteristic Lyapunov exponent of the cycle is bounded below by

$$\frac{1}{n} \sum_{k=0}^{n-1} \ln(\lambda_k) = \frac{1}{n} \ln \left(\prod_{k=0}^{n-1} \lambda_k \right) = \frac{1}{n} \ln(\text{cycle multiplier of } C).$$

Approximating the Morse spectrum of f means computing the characteristic Lyapunov exponent of each simple cycle in Γ , and since there are an exponential number of simple cycles, this is an exponential-time algorithm. Osipenko's test for hyperbolicity says that if the minimum exponent over all simple cycles is positive, then the map is hyperbolic. However, the graph of Example 6.4 also suggests there is no shortcut to efficiently computing the minimum exponent, since taking the log and dividing by the cycle length for this example produces the exact same complications as taking the n^{th} root (the specific numbers can be easily checked).

6.2. Approximate, efficient solutions. In order to efficiently determine a successful box expansion constant L , we need a good starting guess. We have a lower bound of 1 for L . Corollary 1.8 gives a way to get an upper bound: if $\mathcal{M} > 1$ is the smallest cycle multiplier, realized by a cycle of length m , then the ideal box expansion constant (*i.e.*, the smallest average multiplier) \mathcal{L} satisfies $1 < \mathcal{L} \leq \mathcal{M}^{1/m}$.

Alternatively, we can get an upper bound for L in a very simple way. For a polynomial map f , the *Lyapunov exponent* λ measures the rate of growth of tangent vectors to J under iteration. A description of the one-variable case is given in [21].

Theorem 6.5 ([2, 10]). *For a polynomial map $f: \mathbb{C} \rightarrow \mathbb{C}$ of degree $d > 1$, the Lyapunov exponent λ satisfies $\lambda \geq \log d$, with equality if and only if J is connected.*

Thus for degree d maps with connected J , $L = d$ is an upper bound.

One straightforward way to obtain a good L in a preset number of steps is basic *bisection*. Keep track of $\text{lo}L$, the most recent working L , and $\text{hi}L$, the most recent failing L . Lower L halfway to $\text{lo}L$ when box expansion fails, and raise it halfway to $\text{hi}L$ when it succeeds. Start with $\text{lo}L = 1$ and $\text{hi}L = d$ or $\mathcal{M}^{1/m}$ if computed.

An alternative to straight bisection is to utilize the information that Algorithm 4.5, **BuildMetric**, already discovers in a test for box expansion. Indeed, if box expansion fails for some L , then we realized in Proposition 1.7 that it is due to a cycle, *badcycle*, with average multiplier L' less than L . But we can easily adapt the algorithm to compute and return the average multiplier of *badcycle*, L' . Then if $L' \leq 1$, we know the map is not box expansive on Γ and we can stop. Otherwise, on the next pass instead of lowering by some arbitrary amount we may simply test by this new average multiplier L' . Better yet, we can test by the minimum of L' and L minus some preset step size δ , in order to prevent increasingly small steps down. We refer to the above procedure as the algorithm **Find-L-Cycles**.

Proposition 6.6. *Let \mathcal{L} be the minimum average multiplier over all simple cycles in the graph Γ . If $2 \geq \mathcal{L} > 1 + \delta$, then **Find-L-Cycles** shows box expansion by some L within δ of \mathcal{L} , in at most $1/\delta$ trials of **BuildMetric**.*

Proof. Since we are looking for an expansion amount in the interval $[1, 2]$, and decrease by at least δ at each step, we perform at most $1/\delta$ attempts. Suppose

one of these attempts is successful. That is, suppose Γ fails to box expand by some $L_0 \leq 2$, and thus outputs an average cycle multiplier of $L_1 < L_0$. Since \mathcal{L} is the minimum, $\mathcal{L} \leq L_1$. Suppose Hypatia verifies successfully box expansion by $L = \min\{L_1, L_0 - \delta\}$. Then by Lemma 5.2 we have $\mathcal{L} \geq L$. Thus we have:

$$L_0 > L_1 \geq \mathcal{L} \geq L = \min\{L_1, L_0 - \delta\}.$$

Thus, either we were lucky and $\mathcal{L} = L_1$ and we have shown box expansion by exactly that amount, or $L_0 > L_1 > \mathcal{L} \geq L_0 - \delta$ and we have shown expansion by L_1 within δ of \mathcal{L} . \square

The best way to control round-off error in the above with interval arithmetic (see Section 3) is to “round down”, since a box expansion constant larger than the average cycle multiplier would fail. That is, set $L_1 = \text{Inf}((\text{Hull}(\lambda_0) \cdots \text{Hull}(\lambda_{n-1}))^{1/n})$.

7. EXAMPLES OF RUNNING HYPATIA FOR POLYNOMIAL MAPS

In this section, we describe the results of applying the algorithms of this paper to some examples for quadratic and cubic polynomials, $P_c(z) = z^2 + c$ and $P_{c,a}(z) = z^3 - 3a^2z + c$, using our implementation in the computer program Hypatia. In particular, in all of the examples of this section, we used the Handicap Hedging Algorithm and the method **Find-L-Cycles** (Section 6.2), to try to produce an $L > 1$ and a box metric for which the constructed Γ was box expansive. Table 1 at the end of the section summarizes the data for all the examples.

All of the computations described in this section were run on a Sun Enterprise E3500 server with 4 processors, each 400MHz UltraSPARC (though the multiprocessor was not used) and 4 GB of RAM. ³

7.1. Producing the box model Γ . First we summarize how we used the box chain construction from [9] to produce box models Γ for these maps. The box chain construction is in fact an iterative process. We began by defining a large box \mathcal{B}_0 in \mathbb{C} such that $J \subset \mathcal{B}_0$. Then for some $n > 1$, we place a $2^n \times 2^n$ grid of boxes on \mathcal{B}_0 . The construction then builds a graph Γ consisting of a subcollection of these grid boxes, which is a box model of J , according to Definition 1.1. If the boxes are sufficiently small, then every invariant set disjoint from J is disjoint from $\mathcal{B}(\Gamma)$ (for example, attracting periodic orbits). Further, we can produce an improvement of Γ by subdividing all (or some) of the boxes in Γ , then repeating the construction to produce a new graph Γ' such that $\mathcal{B}(\Gamma') \subset \mathcal{B}(\Gamma)$.

For quadratic polynomials, it is easy to check that if $|c| < 2$, then the filled Julia set is strictly contained in the box $[-2, 2]^2$. For the cubic polynomials $P_{c,a}(z) = z^3 - 3a^2z + c$, one can also check that the filled Julia set is contained in $[-2, 2]^2$ whenever $|c| < 2$ and $|a| < (2/3)^{1/2}$. We denote $\mathbb{D}_2 = [-2, 2]^2$. In each of our examples, we began the box chain construction with some $2^n \times 2^n$ grid on $\mathcal{B}_0 = \mathbb{D}_2$.

7.2. Selective subdivision. We also introduce here two modifications to the basic box chain construction, in order to improve efficiency. The key idea for both is *selective subdivision*, i.e., rather than subdividing all of the boxes to form the next level, we only subdivide the boxes where the dynamics is behaving badly.

Thus we must determine which boxes are most obstructing the construction of a hyperbolic metric. One approach is to choose boxes which are closest to a sink

³The server was obtained by the Cornell University mathematics department through an NSF SCREMS grant.

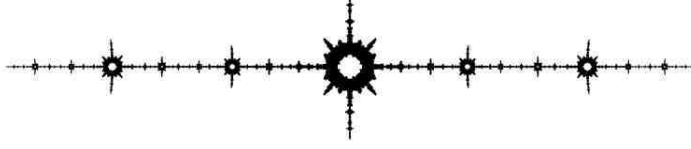


FIGURE 2. A box Julia set for $\mathcal{B}(\Gamma)$, for the map $z \mapsto z^2 - 1.755$, and boxes from a $2^{11} \times 2^{11}$ grid on \mathbb{D}_2 . Γ is box expansive.

cycle, say boxes which have bounded midpoints after several images, and subdivide only those boxes. We call this procedure *sink basin subdivision*. See Example 7.4.

Alternatively, the **Find-L-Cycles** algorithm, in addition to improving the metric on the given graph, can also be used to determine a selective subdivision procedure, which we call *weak cycle subdivision*. After **Find-L-Cycles** is run, we can easily identify the boxes involved in the cycle with the weakest cycle multiplier. Then subdividing these boxes should help improve the expansion amounts. We utilize weak cycle subdivision in Examples 7.2 and 7.3.

7.3. Examples for quadratic polynomials. The quadratic polynomial $P_c(z) = z^2 + c$ has one critical point, at zero, thus has at most one (finite) attracting cycle.

Example 7.1. The aeroplane, P_c for $c = -1.755$, has a period 3 sink. Shown in Figure 2 is a box Julia set $\mathcal{B}(\Gamma)$ consisting of boxes from $2^{11} \times 2^{11}$ grid on \mathbb{D}_2 . This box chain model Γ has 43,000 vertices and 260,000 edges. Γ is box expansive, with box expansion constant $L = 1.05069$. The computation took less than 200 MB of RAM and 6.5 minutes.

Example 7.2. The quadratic polynomial $P(z) = z^2 - 1$ is called the basilica. This map has a period 2 attracting cycle $0 \leftrightarrow -1$. The picture on the upper left of Figure 3 is a heuristic sketch of J , drawn using the program Fractalasm (available at [1]). Shading is according to rate of escape to infinity, or to the two-cycle.

The picture on the lower left of Figure 3 is a box Julia set $\mathcal{B}(\Gamma)$, composed of selected boxes from a $2^7 \times 2^7$ grid on \mathbb{D}_2 . Γ has 1,800 vertices and 14,000 edges. We could have easily made a finer picture, but this rough box Julia set is enough to prove hyperbolicity. Indeed, this box chain model Γ is box expansive, with box expansion constant $L = 1.14067$. The associated box metric has handicaps, φ_k , with minimum 0.019, average 0.049, and maximum 1. This initial computation took only 8 MB of RAM and less than a minute of CPU time.

For a deeper understanding of this metric, we created a picture of the box Julia set, with shading of each box according to the handicap for that box, shown on the lower left of Figure 3. Boxes with smaller handicaps are shaded darker.

Example 7.3. A Cantor Julia set near the cusp of the Mandelbrot set is $c = .35$. The upper right of Figure 3 is a Fractalasm sketch of this Julia set. This is an example in which we can use weak cycle subdivision to show hyperbolicity more quickly. The box chain model from the $2^7 \times 2^7$ grid on \mathbb{D}_2 fails to be box expansive by $L = 1$ due to a bad cycle of length only 1. We had the program subdivide just that box, and again the resulting box chain model failed for $L = 1$ due to a length 1 bad cycle. But, upon subdividing that one box, we found that the resulting box

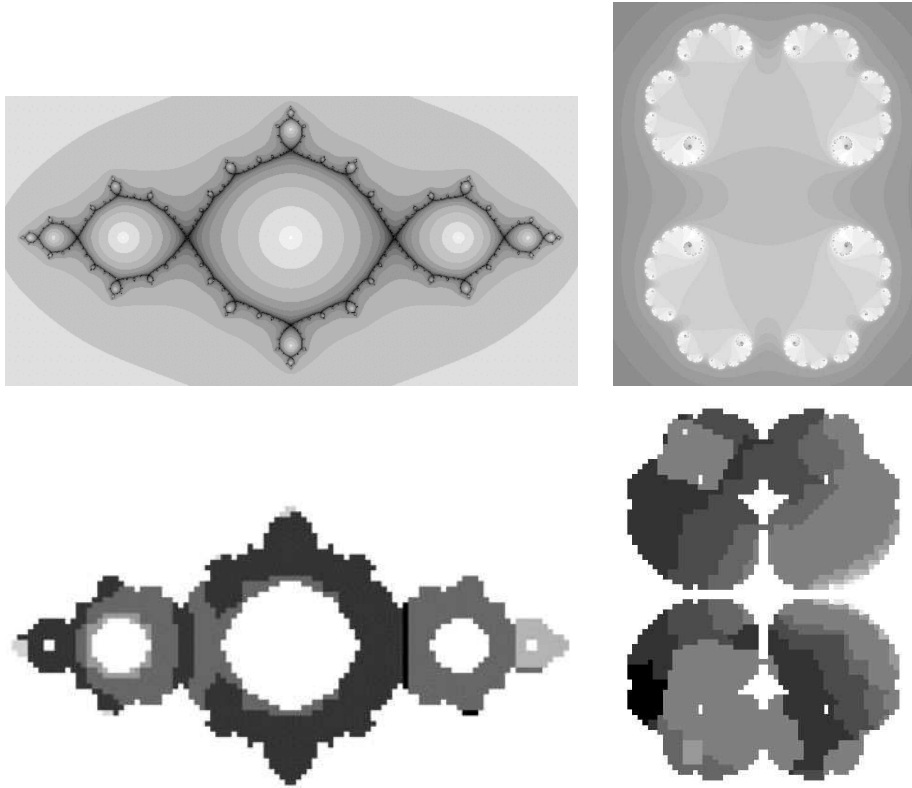


FIGURE 3. *Left:* sketches of J for the map $z \mapsto z^2 - 1$; *Right:* sketches of the cantor J for the map $z \mapsto z^2 + .35$; *Upper:* heuristic sketches of J , drawn with the program Fractalasm; *Lower:* a box-expansive box Julia set $\mathcal{B}(\Gamma)$ for each map, where shading is determined by handicaps, with smaller handicaps colored darker. Note the asymmetry in metric shades. This is not dynamically significant, but an artifact of the tree construction for assigning handicaps. Boxes on the *lower left* have side length $4/2^7$, and boxes on the *lower right* have side length $4/2^8$.

chain model is box expansive (by $L = 1.00778$). On the lower right of Figure 3 is the hyperbolic box Julia set, with shading according to the value of the handicap in each box. Boxes with darker shading have handicaps closer to zero.

7.4. Examples for cubic polynomials. The cubics $P_{a,c}(z) = z^3 - 3a^2z + c$ have two critical points, at $\pm a$, thus have at most two attracting cycles. When $a = 0$, there is a clear correspondence between $z^2 + c$ and $z^3 + c$.

Example 7.4. A seemingly rabbit-like cubic is $P_{a,c}, c = -.44 - .525i, a = .3i$. This map has an attracting period three cycle. Shown in Figure 4 on the left is the box Julia set $\mathcal{B}(\Gamma_{a,c})$ with boxes from a $2^{10} \times 2^{10}$ grid on \mathbb{D}_2 . However, this Γ is not hyperbolic because \mathcal{B} contains one of the critical points, $a = .3i$. We then

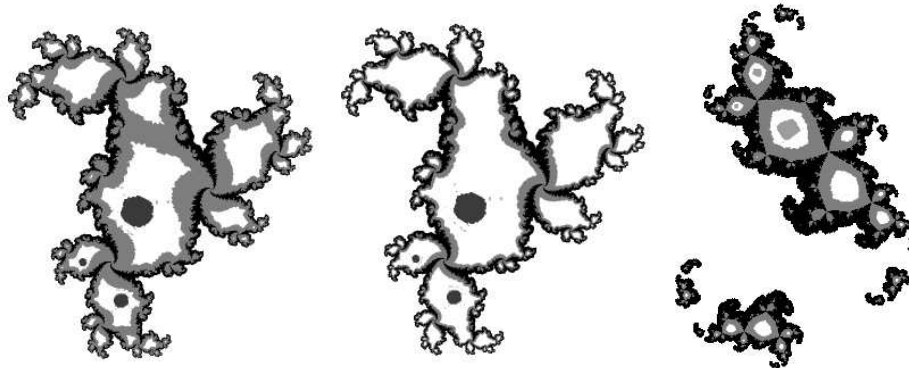


FIGURE 4. Each image above is a box model of J and the period 3 sink for a cubic polynomial $P_{a,c}$. Points heuristically found to be in the filled Julia set are shaded lighter, to illustrate J . *Left:* the box Julia set \mathcal{B} for $c = -.44 - .525i$, $a = .3i$, for a $2^{10} \times 2^{10}$ grid on \mathbb{D}_2 contains a critical point, so the box chain model Γ is not box expansive. *Center:* a refinement of the left picture, with boxes of side length $4/2^{10}$ and $4/2^{11}$, is box expansive. *Right:* the box Julia set \mathcal{B} for the map $c = -.38125 + .40625i$, $a = 0.5i$, with boxes of side length $4/2^9$ and $4/2^{10}$ is box expansive.

performed sink basin subdivision and produced a box chain model Γ' which is box expansive. The latter box Julia set \mathcal{B}' is shown in the center of Figure 4.

Example 7.5. The cubic polynomial $P_{a,c}$, with $c = -.38125 + .40625i$, $a = 0.5i$ also has an attracting cycle of period three, but here the Julia set is disconnected. The most efficient method to get a hyperbolic box chain model Γ is to first subdivide all boxes uniformly to obtain a $2^9 \times 2^9$ grid on \mathbb{D}_2 , and then perform sink basin subdivision. Shown on the right in Figure 4 is the resulting box Julia set.

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TABLE 1. Data for the box chain models described in Section 7. Here the box depth n is the number such that the boxes are of size $4/2^n$ (from a $(2^n \times 2^n)$ grid on \mathbb{D}_2).

Examples for the map $P_c(z) = z^2 + c$

Example	param c	sink period	Figure	box depth n	# Γ boxes (1,000s)	# Γ edges (1,000s)	box- exp?	L	max φ	min φ	avg φ
7.1	-1.755	3	2	11	41	250	Yes	1.0507	1	2.023×10^{-5}	6.8×10^{-5}
7.3	.35	N/A		7	3.028	27.376	No				
			3 R	7, 8, 9	3.034	27.422	Yes	1.00778	1.19	0.0288723	0.119515
7.2	-1	2	3 L	7	1.8	14	Yes	1.14067	1	0.019	0.049

Examples for the map $P_{a,c}(z) = z^3 - 3a^2z + c$

Example	params c a		sink period	Figure	box depth n	# Γ' boxes (1,000s)	# Γ' edges (1,000s)	box- exp?	L	max φ	min φ	avg φ
7.4	-.44	.3i	3	4 L	10	92	142	No				
	-.525i			4 C	10, 11	160	250	Yes	1.0578	1	1.023×10^{-4}	3.5×10^{-4}
7.5	-.38125 +.40625i	0.5i	3	4 R	9, 10	45	867	Yes	1.0369	1	5.08×10^{-4}	.01748

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